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THE STRESSED STATE NEAR A MICROFLAW CLUSTER POINT†

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The stressed state near cluster point $z = 0$ for microflaws (MiFs) in the form of cracks or thin linear sharply-angled inclusions in an elastic plane located along a line on one side or other of the cluster point and satisfy certain conditions is investigated. This is preceded by the analytic solution of the first and second fundamental problems of the theory of elasticity for a plane with an infinite set of collinear linear singularities clustering at a finite point. Cases are considered in which the flaws are such that their images under the mapping $\zeta = 1/z$ are situated periodically along an entire line or only along a ray. Stability to fracture, both in the neighbourhood of an MiF cluster point and globally for an MiF system, are investigated using force and energy fracture criteria. Examples describing the fracture mechanism are given. An analytic solution of the problem of the interaction of a macroflaw (MaF) with an infinite series of MiFs collinear with it and clustering at the vertex of the MaF is obtained. The investigation is based on a conformal mapping and the results of [1–4], in which solutions are obtained in closed form of the first and second fundamental problems of the theory of elasticity for a plane with a denumerable set of cuts with a cluster point at infinity.

There have been previous investigations [5, 6] of the stressed state near a cluster point of microcracks arranged on a logarithmic scale along a ray, and asymptotic representations have been obtained for the stresses and stress intensity factors near the microcrack cluster point. The case considered in [5, 6] differs from the case considered here both in the method of investigation and in the way mechanical fracture is viewed. The problem of the interaction of a macrocrack with an infinite set of microcracks arranged according to a certain law and clustering at infinity has been studied by various methods in [4, 7–9] and elsewhere.

1. PROBLEMS OF THE THEORY OF ELASTICITY FOR A PLANE WITH A DENUMERABLE SET OF LINEAR SINGULARITIES CLUSTERING AT A FINITE POINT

Suppose that a homogeneous isotropic elastic plane with complex variable $b = x + iy$ is weakened by a denumerable set of microflaws (MiFs) in the form of cracks or thin rigid linear sharply-angled inclusions located along sections $L_n = [a_n, b_n]$, $a_n \neq 0$, $b_n \neq 0$, $n \in I$ of the real axis, clustering at the point $z = 0$ and satisfying the conditions

$$|b_{n+1}^{-1} - b_n^{-1}| \geqslant d > 0, \quad a_n^{-1} - b_n^{-1} \leqslant l \tag{1.1}$$

when n is large.

If the intervals cluster at the point $z = 0$ from both sides, then the set of indices $I = \{0; \pm 1; \pm 2; \ldots\}$, where all the intervals with non-negative indices are situated to the right of the cluster point, and all the intervals with negative indices are situated to the left of this point (Fig. 1a). If, however, the intervals cluster only one side (say, the right), then $I = \{0; 1; \ldots\}$, it being possible for a finite set of intervals to be situated to the left of the cluster point (Fig. 1b).

Conditions (1.1) are satisfied if, for example, all but a finite number of the intervals L_n are such that their images under the mapping $\zeta = 1/z$ form a periodic set along the entire real axis or semi-axis. We note that if the MiFs are located log-periodically, as in [6], then the second inequality in (1.1) is not satisfied.

Suppose that on the sides $L_n^{\pm}$ of the MiFs we have specified either the normal stress $\sigma_y^{\pm}(t)$ and the shear stress $\tau_{xy}^{\pm}(t)$ (the first problem) or the partial derivatives $(u', v')^{\pm}(t)$ with respect to x of the

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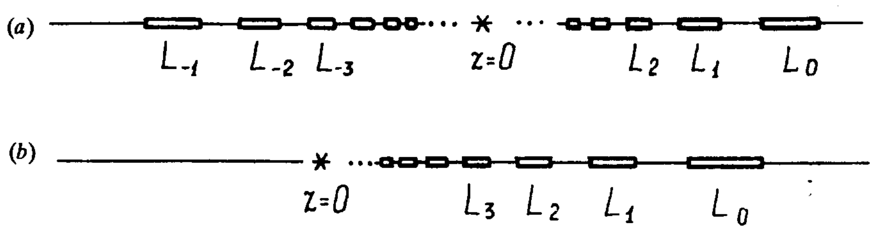

Fig. 1.

displacement components (the second problem), which are Hölder continuous in each interval and can grow no faster than $M\,|\,t\,|^{-\lambda_0}$, $0 \leqslant \lambda_0 < 1$ as $t \to 0$, and that the stresses σ_x^∞, σ_y^∞, τ_{xy}^∞ and rotation ω^∞ are specified at infinity. Furthermore, in the second problem we shall take as given the principal vectors $P_n = X_n + iY_n$ of the external loads acting on the micro-inclusions L_n, diminishing as $n \to \infty$ no slower than $M\,|\,a_n\,|^{2-\lambda_0}$.

Problem. It is required to find the plain stress–strain state outside the flaws L_n, $n \in I$ characterized by finite elastic potential energy both in the neighbourhoods of the vertices of each flaw and in a neighbourhood of the cluster point $z = 0$ from which sufficiently small neighbourhoods of each interval L_n have been removed.

We define neighbourhoods of the intervals L_n as follows. Let L_n^* be the image of the interval L_n under the mapping $\zeta = 1/z$, and let $U_\varepsilon^*(L_n^*)$ be its small ε-neighbourhood, i.e. the set of points in the plane whose distance from L_n^* does not exceed ε. The pre-image of the neighbourhood $U_\varepsilon^*(L_n^*)$ under the inverse transformation $z = 1/\zeta$ is denoted by $U_\varepsilon(L_n)$ and will be called the ε-neighbourhood of the interval L_n; we will call the set of all these neighbourhoods $U_\varepsilon(L)$ and will take it to be the ε-neighbourhood of the "line" L which consists of all the intervals L_n, $n \in I$.

In this case the stress–strain state of the plane with MiFs L_n, $n \in I$ possessing the properties described above is determined by the well-known Kolosov–Muskhelishvili formulae [10] in terms of complex potentials $\Phi(z)$, $\Omega(z)$ which at the ends of the intervals L_n can become infinite with order less than one, while for small z, situated outside any fixed small neighbourhood $U_\varepsilon(L)$ of the line L understood as above, these potentials do not exceed $M\,|\,z\,|^{-\lambda}$ in modulus for some $\lambda < 1$. On the line L they satisfy the boundary conditions

$$\rho\Phi^+(t) + \Omega^-(t) = f^+(t), \quad \rho\Phi^-(t) + \Omega^+(t) = f^-(t), \quad t \in L \tag{1.2}$$

where in the case of the first problem $\rho = 1, f^\pm(t) = (\sigma_y - i\tau_{xy})^\pm$ and in the case of the second problem $\rho = -\kappa; f^\pm(t) = -2\mu(u' + iv')^\pm$ κ and μ are the elasticity constants of the material. In a neighbourhood of infinity these functions have the form

$$\Phi(z) = \Gamma - \frac{P}{2\pi(\kappa+1)}\frac{1}{z} + O(z^{-2}), \quad \Omega(z) = \Gamma' + \frac{\kappa P}{2\pi(\kappa+1)}\frac{1}{z} + O(z^{-2}) \tag{1.3}$$

$$\Gamma = \frac{1}{4}(\sigma_x^\infty + \sigma_y^\infty) + i\frac{2\mu}{\kappa+1}\omega^\infty, \quad \Gamma' = \sigma_y^\infty - i\tau_{xy}^\infty - \Gamma \tag{1.4}$$

where P is the principal vector of the external loads acting on all the flaws.

Adding and subtracting conditions (1.2) from one another, we obtain the boundary conditions

$$\begin{gathered} \Phi_1^+(t) + \Phi_1^-(t) = 2p(t), \quad \Phi_2^+(t) - \Phi_2^-(t) = 2q(t), \quad t \in L \\ 2p(t) = f^+(t) + f^-(t), \quad 2q(t) = f^+(t) - f^-(t) \end{gathered} \tag{1.5}$$

for finding the functions $\Phi_{1,2}(z) = \rho\Phi(z) \pm \Omega(z)$, and the denumerable set of conditions

$$\int_{L_n} (\Phi_1^+(t) - p(t))dt = \frac{i(\rho-\kappa)}{2(\kappa+1)}P_n, \quad n \in I \tag{1.6}$$

where P_n is the principal vector of external loads acting on the flaw L_n. In the case of the first problem conditions (1.6) express the single-valuedness of the displacements after making a circuit around an MiF. These functions have all the properties of the function $\Phi(z)$ at the ends of the intervals L_n and in a neighbourhood of the cluster point $z = 0$, while in a neighbourhood of infinity they have the form

$$\begin{gathered}\Phi_k(z) = B_k + C_k z^{-1} + O(z^{-2}), \quad k = 1,2 \\ B_1 = (\rho - 1)\Gamma + \sigma_y^\infty - i\tau_{xy}^\infty, \quad B_2 = (\rho + 1)\Gamma - \sigma_y^\infty + i\tau_{xy}^\infty, \\ C_1 = \frac{(\kappa - \rho)P}{2\pi(\kappa + 1)}, \quad C_2 = -\frac{(\kappa + \rho)P}{2\pi(\kappa + 1)}\end{gathered} \tag{1.7}$$

In terms of the functions $\Phi_{1,2}(z)$ the Kolosov–Muskhelishvili formulae can be written in the form

$$\begin{gathered}\rho(\sigma_x + \sigma_y) = 2\,\mathrm{Re}(\Phi_1(z) + \Phi_2(z)), \quad 4\mu\rho\omega = (\kappa + 1)\,\mathrm{Im}(\Phi_1(z) + \Phi_2(z)) \\ 2\rho(\sigma_y - i\tau_{xy}) = \Phi_1(z) + \Phi_2(z) + \rho\Phi_1(\bar{z}) - \rho\Phi_2(\bar{z}) + (z - \bar{z})(\overline{\Phi_1'(z)} + \overline{\Phi_2'(z)}) \\ 4\mu\rho\frac{\partial}{\partial x}(u + iv) = \kappa\Phi_1(z) + \kappa\Phi_2(z) - \rho\Phi_1(\bar{z}) + \rho\Phi_2(\bar{z}) - (z - \bar{z})(\overline{\Phi_1'(z)} + \overline{\Phi_2'(z)})\end{gathered} \tag{1.8}$$

where in the case of the first problem $\rho = 1$ and for the second $\rho = -\kappa$.

2. SOLUTION OF THE PROBLEM

We apply the conformal mapping $\zeta = 1/z$ to the elastic domain. The line L then becomes the line L^* consisting of intervals $L_n^* = \{\alpha_n, \beta_n\}$, $\alpha_n = 1/b_n$, $\beta_n = 1/a_n$, $n \in I$ which cluster at infinity and according to (1.1) satisfy the conditions $\alpha_{n+1} - \alpha_n \geqslant d$, $\beta_n - \alpha_n \leqslant l$ for large n. Then the functions

$$\Psi_{1,2}(\zeta) = \zeta^{-2}\Phi_{1,2}(1/\zeta) = z^2\Phi_{1,2}(z) \tag{2.1}$$

satisfy all the conditions of [1] together with the additional condition

$$\Psi_k(\zeta) = B_k\zeta^{-2} + C_k\zeta^{-1} + O(1), \quad k = 1, 2$$

which should be satisfied in a neighbourhood of the point $\zeta = 0$. Writing out the functions $\Psi_{1,2}(\zeta)$ and changing back from ζ to z, we find that

$$\Phi_1(z) = \mathrm{X}(z)(R(z) + Q(z) + \alpha B_1), \quad \alpha^{-1} = \lim_{z\to\infty} \mathrm{X}(z) = \prod_{n\in I} \frac{\sqrt{a_n b_n}}{c_n} \tag{2.2}$$

$$\mathrm{X}(z) = \prod_{n\in I} \frac{\sqrt{a_n b_n}}{c_n} \frac{z - c_n}{\sqrt{(z - a_n)(z - b_n)}}, \quad c_n = \frac{2a_n b_n}{a_n + b_n} \tag{2.3}$$

$$R(z) = \sum_{n\in I} \frac{1}{\pi i(z - c_n)} \int_{L_n} \frac{t - c_n}{\mathrm{X}^+(t)} \frac{p(t)dt}{t - z} \tag{2.4}$$

$$Q(z) = \sum_{n\in I} A_n / (z - c_n) \tag{2.5}$$

$$\Phi_2(z) = B_2 + \frac{1}{\pi i} \int_L \frac{q(t)dt}{t - z}, \quad L = \bigcup_{n\in I} L_n \tag{2.6}$$

The infinite product (2.3) and series (2.4) converge absolutely and uniformly in any closed bounded domain which does not contain points of the line L. Each square root in the product (2.3) denotes that branch of the multivalued function which is single-valued in the plane with the corresponding cut $[a_n, b_n]$ and which becomes equal to z as $z \to \infty$. The constants $B_{1,2}$ and functions $p(t)$, $q(t)$ are found from formulae (1.7), (1.4) and (1.5), respectively, while the constants A_n, $n \in I$ are found from the infinite series of linear algebraic equations

$$\sum_{k\in I} \delta_{nk} A_k = H_n, \quad n \in I \tag{2.7}$$

$$\delta_{nk} = \int_{L_n} \frac{X^+(t)}{t-c_n} dt, \quad H_n = \frac{i(\rho-\kappa)}{2(\kappa+1)} P_n - \int_{L_n} (\alpha B_1 + R(t)) X^+(t) dt \tag{2.8}$$

The solution of this system has to be found in the class of those complex sequences A_k, $k \in I$ such that in any closed bounded domain not containing the points c_n, $n \in I$ the series (2.5) converges uniformly and such that for small $z \notin U_\varepsilon(L)$ its sum $Q(z)$ does not exceed $M|z|^{-\lambda}$ in modulus for some $\lambda < 1$. System (2.7) is a consequence of conditions (1.6). If this system is solvable in the given class of sequences, its solution is unique and can be found by the reduction method or the method of successive approximations. Our later arguments assume that system (2.7) is solvable. Some cases when this system is solvable will be considered in Sections 4 and 5.

In particular, if the boundary conditions of the problem are zero, and if in the second problem the principal vectors P_n are also zero, then

$$\Phi_1(z) = \alpha B_1 X(z)(1 - \sum_{n\in I} A_n / (z - c_n)), \quad \Phi_2(z) = B_2 \tag{2.9}$$

The constants A_n are found from system (2.7) where $H_n = \int_{L_n} X^+(t) dt$.

3. THE BEHAVIOUR OF THE STRESSES AND STRESS INTENSITY FACTORS IN THE NEIGHBOURHOOD OF A MICROFLAW CLUSTER POINT

From the results of [1, 11] for the behaviour of the function $\Psi_2(\zeta)$ for large ζ and Eq. (2.1), it follows that for small z situated outside any fixed small ε-neighbourhood $U_\varepsilon(L)$ of the line L (as explained in Section 1) the function $\Phi_2(z)$ does not exceed $M|z|^{-\lambda}$ in modulus for any $\lambda \in (\lambda_0, 1)$, where λ_0 describes the rate of growth of the original data of the problem near the MiF cluster point. The function $R(z)$ does not exceed $M|z|^{-\lambda}$ in modulus. Because the function $X(z)$ is bounded outside the neighbourhood $U_\varepsilon(L)$, the behaviour of the function $\Phi_1(z)$ for small $z \notin U_\varepsilon(L)$ also depends on the nature of the function $Q(z)$, which does not exceed $M|z|^{-\lambda_1}$ in modulus, where λ_1 is some non-negative number less than unity. Consequently, if $\lambda_1 > \lambda_0$, then for small $z \notin U_\varepsilon(L)$ the function $\Phi_1(z)$ does not exceed $M|z|^{-\lambda}$ in modulus, and according to (1.8) the stresses, rotation and derivatives of the displacement components for small $z \notin U_\varepsilon(L)$ also do not exceed $M|z|^{-\lambda}$ in modulus. If $\lambda_1 \leq \lambda_0$, they do not exceed $M|z|^{-\lambda}$ in modulus for any $\lambda \in (\lambda_0, 1)$.

When $z \to 0$ along any fixed ray in the upper or lower half-plane, the functions $X(z)$ and $R(z)$ in formula (2.2) tend to the limits 1 and 0, respectively, while the function $\Phi_2(z)$ does not exceed $M|z|^{-\lambda}$ in modulus when $0 < \lambda_0 < 1$ and $M \ln|z|^{-1}$ when $\lambda_0 = 0$. The behaviour of the stresses along these rays is therefore governed by the behaviour of the function $Q(z)$ as $z \to 0$ along these rays, which in turn depends on the behaviour of the solution of system (2.7) when $n \to \infty$.

We will find the stress intensity factors near the vertex $g_n = a_n$ or $g_n = b_n$ of the flaw L_n [12, 13]

$$K_1(g_n) - iK_2(g_n) = \lim_{x\to g_n, x\notin L} \frac{2\rho}{\rho+1} \sqrt{2\pi|x-g_n|}(\sigma_y(x+i0) - i\tau_{xy}(x+i0))$$

where in the case of the first problem $\rho = 1$ and in the second $\rho = -\kappa$. From formulae (1.8), (2.2)–(2.6) we find that

$$K_1(g_n) - iK_2(g_n) = \lim_{x\to g_n, x\notin L} \sqrt{2\pi|x-g_n|}\Phi_1(x+i0) = \eta(g_n)[R(g_n) + Q(g_n) + \alpha B_1] \tag{3.1}$$

$$\eta(g_n) = g_n(a_n+b_n)^{-1}\alpha^{-1}\sqrt{2\pi(b_n-a_n)} \prod_{k\in I, k\neq n} (g_n - c_k) / \sqrt{(g_n-a_k)(g_n-b_k)} \tag{3.2}$$

According to (1.1) the inequalities

$$b_n - a_n \leqslant l a_n b_n \leqslant M g_n^2, \quad |g_n| \leqslant M|n|^{-1}$$

are satisfied for large n, while the functions $R(g_n)$, $Q(g_n)$ can increase no faster than $n \to \infty$ as $M|z|^{-\lambda}$, and so the stress intensity factors satisfy the inequality

$$|K_{1,2}(g_n)| \leqslant M|g_n|^{1-\lambda} \leqslant M|n|^{\lambda-1}, \quad 0 \leqslant \lambda < 1$$

for large n, i.e. they become as small as desired for flaw vertices situated sufficiently close to the cluster point. Hence, in terms of the force fracture criteria [14] the system of MiFs L_n, $n \in I$ we are considering is always stable with respect to fracture in some neighbourhood of the MiF cluster point. In the case of global instability with respect to fracture, the fracture begins at the vertices of some finite set of flaws that are "far" from the cluster point and depend on the external loads and distribution of the MiFs. In this way the force criterion mechanism of the fracture of the MiF system under consideration differs from the mechanism of the fracture of the system of microcracks that has been previously considered [6]. The latter is unstable to fracture for arbitrarily small external loads, and its fracture occurs in a neighbourhood of the MiF cluster point through the merging of all but a finite number of cracks. Specific examples of fractures that occur through the mechanism we described above will be given in the following section.

We also consider the stability of the MiF cluster point $z = 0$ to fracture using the energy criterion, and to this end we study the invariant complex Γ-integral of Rice–Cherepanov [15] along a circle of small radius r centred at the point $z = 0$. If this circle intersects a sufficiently small neighbourhood of some flaw L_n in the sense of Section 1, then the part of the circle which lies within this neighbourhood is replaced by the smaller part of the boundary of the neighbourhood.

Suppose that system (2.7) is solvable and that for large n its solution satisfies the inequality $|A_n| \leqslant M_1|a_n|^{2-\lambda_1}$, $0 \leqslant \lambda_1 < 1$. Then $Q(z)$ does not exceed $z \to 0$ in modulus when $0 < \lambda_1 < 1$ and $M_2 \ln|z|^{-1}$ when $\lambda_1 = 0$ as $M|z|^{-1}$ along any fixed ray in the upper or lower half-plane, but for small $z \notin U_\varepsilon(L)$ one can only assert that $|Q(z)| \leqslant M_3|z|^{-\lambda}$ for any $\lambda \in (\lambda_1, 1)$. The function $\Phi_2(z)$ has the same properties as $z \to 0$ along the given rays, except that one must now use λ_0 instead of λ_1.

We put $\lambda = \max\{\lambda_0; \lambda_1\}$. Then according to (1.8) the Γ-integral under consideration has the estimate $|\Gamma| \leqslant Mr^{1-2\lambda}$, from which it is clear that in the case $\lambda < 1/2$ this integral decreases without limit as $r \to 0$. Hence, in this case, in the energy criterion approach, the fracture of some small neighbourhood of the MiF cluster point is again stable to fracture and the global fracture of the MiF system will again proceed according to the mechanism described above. Examples where this situation occurs will be given in Section 4.

If $\lambda \geqslant 1/2$, the stability of a neighbourhood of the MiF cluster point to fracture in the energy criterion approach depends on the value of the integral Γ, which as $r \to 0$ can have a definite finite limit or can increase without limit. In each such specific case it is necessary to carry out an additional investigation.

Remark. Using a conformal mapping and the results of [16], one can similarly investigate the stressed state near the finite cluster point of an infinite set of closed microcracks in a piecewise-homogeneous elastic plane which is situated along the contact line of the media. A case was considered in [16] in which the cracks were clustered at infinity.

4. A TWO-SIDED PERIODIC DISTRIBUTION OF MICROFLAWS SEALED ACCORDING TO THE MAPPING $\zeta = 1/z$

Suppose that the flaws are situated in the intervals $L_n = \{a_n, b_n\}$ where $a_n^{-1} = (n + 1/2)T + a$, $b_n^{-1} = (n + 1/2)T - a$, $a < T/2$, $n = 0, \pm 1, \ldots$, i.e. they cluster at the point $z = 0$ from both sides (Fig. 1a), and that their images $L_n = [(n + 1/2)T - a, (n + 1/2)T + a]$ under the mapping $\zeta = 1/z$ form a periodic set with period T lying along the entire real axis.

In this case, according to the results of [2] the function

$$X(z) = \frac{\cos\eta}{\sqrt{\cos(\eta + b)\cos(\eta - b)}}, \quad \eta = \frac{\pi}{Tz}, \quad b = \frac{\pi a}{T} \tag{4.1}$$

while system (2.7) has the form

$$\sum_{k=-\infty}^{\infty} \alpha_{n-k} A_k - \alpha_{n+1/2} \sum_{k=-\infty}^{\infty} A_k = iH_n, \qquad n = 0, \pm 1, \ldots$$

$$\alpha_n = \int_0^b \frac{2x}{x^2 - n^2\pi^2} \frac{\sin x\, dx}{(\sin^2 b - \sin^2 x)^{1/2}}$$

From this, adding all the equations, and summing the resulting series and computing the integrals, we find the sum $\Sigma A_k = C_1 \cos b$ where C_1 is found from formula (1.7). Hence system (2.7) has the form

$$\sum_{k=-\infty}^{\infty} \alpha_{n-k} A_k = G_n, \quad n = 0, \pm 1, \ldots$$
$$G_n = \frac{\kappa - \rho}{2\pi(\kappa + 1)} (\pi P_n + \alpha_{n+1/2} P \cos b) - i \int_{L_n} (R(t) + B_1 \cos b) X^+(t) dt \tag{4.2}$$

For large n the numbers G_n satisfy the inequalities

$$|G_n| \leqslant M_0 |n|^{\lambda_0 - 2}, \quad \lambda_0 < 1$$

and so [17, 18] system (4.2) is solvable in the class of sequences described in Section 2 and has the unique solution

$$A_n = \frac{1}{2\pi i} \int_{|t|=1} \frac{G(t)}{\alpha(t)} \frac{dt}{t^{n+1}} = \sum_{k=-\infty}^{\infty} \xi_{n-k} G_k, \quad G(t) = \sum_{n=-\infty}^{\infty} G_n t^n, \quad \alpha(t) = \sum_{n=-\infty}^{\infty} \alpha_n t^n \tag{4.3}$$

where ξ_n are coefficients of the complex Fourier series of the function $1/\alpha(t)$ in the interval $[0, 2\pi]$. For large n the numbers A_n satisfy the inequality $|A_n| \leqslant M_1 |n|^{\lambda-2}$ for any $\lambda \in (\lambda_0, 1)$. The solution of system (4.2) can also be found by the reduction method.

According to (3.1), (2.2) and (4.1) the stress intensity factors near the vertex $g_n = a_n$ or $g_n = b_n$ obey the formula

$$K_1(g_n) - iK_2(g_n) = |g_n| \sqrt{T \operatorname{tg} b} [R(g_n) + Q(g_n) + B_1 \cos b], \quad b = \pi a / T$$

The functions R, Q and the constant B_1 are found from formulae (2.4), (2.5) and (1.7), (1.4), respectively.

In this case the functions $R(z)$, $Q(z)$, which also means the functions $\Phi_1(z)$, together with the function $\Phi_2(z)$ as $z \to 0$ outside any fixed small neighbourhood $U_\varepsilon(L)$ can increase no faster than $M|z|^{-\lambda}$ for any $\lambda \in (\lambda_0, 1)$, where λ_0 describes the rate of growth of the original data of the problem in the neighbourhood of the point $z = 0$. This follows from the results of [2, 11] and the property of A_n given above. Consequently, the stresses, rotation and the derivatives of the displacement components can increase without limit as $z \to 0$ outside $U_\varepsilon(L)$, but no faster than $M|z|^{-\lambda}$ for any $\lambda \in (\lambda_0, 1)$, while the stress intensity factors $K_{1,2}(g_n)$ decrease no slower than $M|z|^{-\lambda}$ outside $n \to \infty$. In the case $\lambda_0 < 1/2$ the invariant Γ-integral along a circle of radius r with centre at $z = 0$ always tends to zero as $r \to 0$, while in the $\lambda_0 \geqslant 1/2$ case it can have a finite or infinite limit. In particular, if all the initial data of the problem, apart from the loads at infinity, are zero, the solution of the problem is given by functions (2.9), and λ can be taken to be as small as desired.

Example 1. Suppose that the plane is weakened by cracks L_n, $n = 0, \pm 1, \ldots$, acted upon at their edges by constant stresses $\sigma_y^+ = -\sigma_y^- = \sigma_n$, $\tau_{xy}^+ = \tau_n$, while a specified stress and rotation act at infinity. Then the function $R(z) \equiv 0$, while the principal vector of the external forces applied to the edges of the crack L_n is equal to $P_n = 2i(\sigma_n - i\tau_n)(a_n - b_n)$. We take stresses σ_n, τ_n such that

$$\frac{\kappa - \rho}{2(\kappa + 1)} P_n = iB_1 \cos b \int_{L_n} X^+(t) dt \tag{4.4}$$

i.e.

$$\sigma_n - i\tau_n = ia^{-1}(\kappa - 1)^{-1}((2n+1)^2 T^2 - 4a^2) B_1 \beta_n, \quad B_1 = \sigma_y^\infty - i\tau_{xy}^\infty \tag{4.5}$$

$$\beta_n = 4(2n+1)(\kappa+1)\pi^2 T^{-1}\cos b \int_0^b \frac{x}{(4x^2-(2n+1)^2\pi^2)^2}\frac{\sin x\,dx}{(\sin^2 b - \sin^2 x)^{1/2}}$$

Hence, in particular, it is clear that for large n the stresses σ_n, τ_n decrease as $1/n$, while the numbers P_n decrease as $1/n^3$. Then the principal vector $P = \Sigma P_n = 0$ and in system (4.2) all the $G_n = 0$. Hence the solution of the system is trivial and $Q(z) \equiv 0$. The solution of the elasticity theory problem is therefore given by the functions

$$\Phi_1(z) = B_1(\cos b)X(z), \quad \Phi_2(z) = B_2 + \sum_{n=-\infty}^{\infty} \frac{\sigma_n - i\tau_n}{\pi i}\ln\frac{b_n - z}{a_n - z}$$

and the stress intensity factors are found from the equation

$$K_1(g_n) - iK_2(g_n) = |g_n|\sqrt{\tfrac{1}{2}T\sin 2b}\,B_1 \tag{4.6}$$

where $g_n = a_n$ or $g_n = b_n$. From this it is clear that the largest value of $|K_1 - iK_2|$ is reached when $g_n = a_{-1}$ and $g_n = b_0$, while when $n \to \infty$ the stress intensity factors decrease as $1/n$.

Consequently, in the force criterion approach to fracture, when $|K_1 - iK_2|$ reaches a critical value, the fracture of the crack system starts at the left tip a_{-1} of crack L_{-1} and the right tip b_0 of crack L_0, simultaneously and to the same degree. Because the functions $\Phi_{1,2}(z)$ are bounded outside any field small neighbourhood of the line L, the same situation applies in the energy approach to fracture.

Example 2. Suppose that the plane is weakened by thin rigid, rectilinear sharply-angled inclusions L_n, $n = 0, \pm 1, \ldots$, and that specified stresses and rotation are applied at infinity. We apply loads to the inclusions L_n so that Eqs (4.4) are satisfied, i.e. $P_n = 8\kappa^{-1}B_1\beta_n$ where B_1 and β_n are found from formulae (1.7), (1.4) and (4.5), respectively. Then the stress intensity factors again satisfy Eq. (4.6) and the situation described in Example 1 applies to the stability of this system to fracture.

5. A ONE-SIDED PERIODIC DISTRIBUTION OF MICROFLAWS SCALED ACCORDING TO THE MAPPING $\zeta = 1/z$

Suppose that the flaws are distributed along the sections $L_n = [a_n, b_n]$ where $a_n^{-1} = (n + \tfrac{1}{2})T + a$, $b_n^{-1} = (n + \tfrac{1}{2})T - a$, $a < T/2$, $n = 0, 1, \ldots$, i.e. they are always to the right of the cluster point $z = 0$ (Fig. 1b) and their images under the mapping $\zeta = 1/z$ form a periodic set with period T located only on the positive real semi-axis.

In this case we have [3]

$$X(z) = \sqrt{\Gamma(\eta + b)\Gamma(\eta - b)}\,/\,\Gamma(\eta), \quad \eta = \tfrac{1}{2} - 1/(Tz), \quad b = a/T$$

where $\Gamma(z)$ is the Euler gamma-function, and system (2.7), after transformations similar to those applied to this system in Section 5, takes the form

$$\sum_{k=0}^{\infty} \gamma_{nk}A_k = iH_n + \alpha(C_1 + \beta B_1)\gamma_{n,-1/2}, \quad n = 0, 1, \ldots \tag{5.1}$$

$$\gamma_{nk} = \int_{-b}^{b} \frac{|\Gamma(x-n+b)\Gamma(x-n-b)|^{1/2}}{\Gamma(x-n)}\frac{dx}{n-k-x}, \quad b = \frac{a}{T}$$

$$\beta = \sum_{n=0}^{\infty} \frac{8a^2}{T(2n+1)(4a^2-(2n+1)^2T^2)} = \frac{1}{2T}\left[\Psi\left(\frac{1}{2}+b\right) + \psi\left(\frac{1}{2}-b\right) - 2\psi\left(\frac{1}{2}\right)\right]$$

$$\psi(z) = (\ln\Gamma(z))', \quad \alpha = \sqrt{\cos\pi b}$$

The constants C_1, B_1, H_n are found from formulae (1.7), (1.4) and (2.8), and the stress intensity factors obey the formula

$$K_1(g_n) - iK_2(g_n) = |g_n|\,\Gamma(n+1\pm b)\left(\frac{T\,\mathrm{tg}\,\pi b}{n!\,\Gamma(n+1\pm 2b)}\right)^{1/2}(R(g_n) + Q(g_n) + B_1\sqrt{\cos\pi b})$$

where the upper plus sign refers to the vertex $g_n = a_n$ and the lower minus sign to $g_n = b_n$. The unique solvability of system (5.1) was proved in [3], but the solution was not found in explicit form. The solution can be obtained by the reduction method.

All the results and derivations of Section 4 apply to the behaviour of the stresses and stress intensity factors in the neighbourhood of this MiF cluster point.

6. THE INTERACTION OF A MACROFLAW WITH AN INFINITE SERIES OF MICROFLAWS

Suppose that the plane is weakened by a macroflaw (MaF) $L_0 = [-2l, 0]$, $l > 0$ and MiFs $L_n = [a_n, b_n]$, $a_n > 0$, $b_n > 0$, $n = 1, 2, \ldots$ which cluster at the tip $z = 0$ of the MaF (Fig. 2a).

In this case all the results of Section 2 remain true if one puts $I = \{0; 1; \ldots\}$, $a_0 = -2l$, $c_0 = -l$ in all the formulae, and also puts $\sqrt{(a_0 b_0)}/c_0 = 1$ in formulae (2.2) and (2.3). Here the solution of system (2.7) has to be found in the class of sequences A_k, $k, \in I$ such that the sum $Q(z)$ of series (2.5) for small $z \notin U_\varepsilon(L)$ does not exceed in modulus the expression $M\,|\,z\,|^{-\lambda}$ for some $\lambda < 1/2$. For the stress intensity factors at the tips of the MiFs L_n, $n = 1, 2, \ldots$, formulae (3.1) and (3.2) hold with the above stipulations, and the stress intensity factors at the left tip $a_0 = -2l$ of the MaF are found from formula (3.1) where

$$\eta(a_0) = \alpha^{-1}\sqrt{\pi l}\prod_{n=1}^{\infty}(a_0 - c_n)/\sqrt{(a_0 - a_n)(a_0 - b_n)}$$

For the right tip $z = 0$ of the MaF, the neighbourhood of which contains an infinite set of MiFs, it is in general impossible to define a stress intensity factor, because in the general case for small $z \notin U_\varepsilon(L)$ only an upper estimate is known for the functions $\Phi_{1,2}(z)$, and the precise asymptotic forms are unknown. Nevertheless, there are cases when these functions have definite asymptotic forms as $z \to 0$ along certain rays. Then according to (1.8), (2.2)–(2.6), as $z \to 0$ the stresses will also have a definite asymptotic form containing parameters which can be taken to be the stress intensity factors.

For example, suppose that when $z \to 0$ along the imaginary axis the function $Q(z) \sim A(2\pi z)^{-\lambda}$, $0 \leqslant \lambda < 1/2$. Then the same asymptotic form occurs $z \to 0$ along any fixed ray lying in the upper or lower half-plane. Here the function $z \to 0$ defined by formula (2.4) has the limit $B = R(0)$ and

$$\Phi_1(z) \sim (K_1 - iK_2)(2\pi z)^{-\lambda-1/2}, \quad K_1 - iK_2 = A\sqrt{\pi l}, \quad 0 < \lambda < 1/2 \tag{6.1}$$

$$\Phi_1(z) \sim (K_1 - iK_2)(2\pi z)^{-1/2}, \quad K_1 - iK_2 = (A + B + \alpha B_1)\sqrt{\pi l}, \quad \lambda = 0 \tag{6.2}$$

Branches of multivalued functions in the plane with a cut along the negative real semi-axis are taken so that on the positive real semi-axis they take real positive values For the function $\Phi_2(z)$ given by formula (2.6), when $z \to 0$ along a given ray we have the estimate $|\,\Phi_2(z)\,| \leqslant M\,|\,z\,|^{-\lambda}$ when $0 < \lambda < 1$ and $|\,\Phi_2(z)\,| \leqslant M \ln |\,z\,|^{-1}$ when $\lambda_0 = 0$, where λ_0 describes the growth rate of the original data of the problem near the point $z = 0$.

Suppose $\lambda_0 < \lambda + 1/2$. Then according to (1.8) the asymptotic form of the stresses as $z \to 0$ is completely determined by the representations (6.1) and (6.2), and the constants K_1 and K_2 in these representations are naturally taken to be the stress intensity factors at the tip $z = 0$ of the MaF. When $\lambda = 0$ the stress intensity factor in this sense and the stress intensity factor in the classical sense [12, 13] are identical. In the case $\lambda_0 \geqslant \lambda + 1/2$ the asymptotic form of the stresses as $z \to 0$ also depends on the behaviour of the function $\Phi_2(z)$ as $z \to 0$. In each case one has to perform additional investigations, which we shall not dwell on.

If the MaF has the asymptotic form (6.1) near to the tip $z = 0$, the invariant Rice–Cherepanov Γ-integral computed along a circle of small radius r and centre at $z = 0$ increases as $r \to 0$ as $r^{-2\lambda}$. This indicates the instability of the MaF to fracture in terms of the energy criterion.

If the MaF has the asymptotic form (6.2) near to $z = 0$, then in the case when $\lambda_0 \leqslant 1/2$ the given Γ-integral will have a finite limit as $L_0 = (-\infty, 0]$ whose value determines the stability of the tip of the MaF to fracture. In the case when $\lambda_0 > 1/2$ the Γ-integral as $r \to 0$ can have both finite and infinite limits. In each such case an additional investigation is required.

Suppose that the MaF lies along the ray $L_0 = (-\infty, 0]$ (Fig. 2b) and the boundary conditions specified along its sides decrease no slower than $M\,|\,t\,|^{-\lambda_0}$, $\lambda_0 > 1/2$ as $t \to \infty$. In this case all the results of Section 2 still remain true if the functions $X(z)$ and $R(z)$ are replaced by the functions

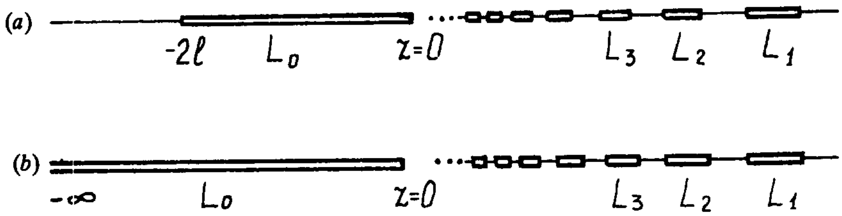

Fig. 2.

$$X_1(z) = X(z)/\sqrt{z}, \quad R_1(z) = R(z) + \frac{1}{\pi i}\int_{-\infty}^{0}\frac{p(t)}{X_1^+(t)}\frac{dt}{t-z}$$

and $I = \{1; 2; \ldots\}$, $L = L_0 \cup L_1 \cup \ldots$ are taken in all formulae, while the solution of system (2.7) is sought in the class of sequences A_k, $k \in I$ such that the sum $Q(z)$ of series (2.5) for small $z \in U_\varepsilon(L)$ does not exceed $M \mid z \mid^{-\lambda}$ in modulus for some $\lambda < 1/2$. Here, according to (1.8), in order to find the constant B_2 in formulae (2.6) in the case of the first problem (i.e. when $\rho = 1$) it is necessary to specify the value of the hydrostatic stress $(\sigma_x^\infty + \sigma_y^\infty)/2$ at infinity and the rotation ω^∞, while in the case of the second problem $\rho = -\kappa$ we need the value $(\sigma_x^\infty + \sigma_y^\infty)/2$ or only σ_y^∞ and ω^∞ or τ_{xy}^∞. Because the function $\Phi_1(z) \sim B_1 z^{-1/2}$ for large z outside some fixed neighbourhood of the ray L_0, in order to find the constant B_1 in formula (2.2) it is necessary to specify not just the values of these parameters as $z \to \infty$, but also their asymptotic form with accuracy to $\mid z \mid^{-1/2}$ inclusive.

In this case formulae (3.1) and (3.2) also hold for the stress intensity factors at the tips of the MiFs if one takes $I = \{1; 2; \ldots\}$ and the number $\eta(g_n)$ is divided by $\sqrt{(g_n)}$, while the situation previously described again applies to the problem of the stress intensity factor at the tip $z = 0$ of the MaF.

The model considered above of an elastic plane with an infinite set of MiFs in the form of cracks or thin rigid linear inclusions which cluster at a finite point can be used to study the stressed state near a point with a small neighbourhood containing a large number of MiFs of the given type, situated in a given way and strongly concentrated near that point. In this case a set of MiFs, which must in reality be finite, can be replaced by an infinite set of MiFs containing new flaws such that to a given accuracy they reflect the order and nature of the positioning of the original flaws. One can also use other models to describe the actual object. For example, the neighbourhood with flaws can be replaced by a material without flaws and described by different elasticity constants. To determine the effective elasticity constants of the new material one can use the model described above.

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